

On discrete features of the wave equation in singular pp-wave backgrounds

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ABSTRACT: We analyze the wave equation in families of pp-wave geometries developing strong localized scale-invariant singularities in certain limits. For both cases of well-localized pp-waves and the so-called null-cosmologies, we observe an intriguing discrete dependence of the existence of a singular limit on the normalization of the pp-wave profile. We also find restrictive matching conditions relating the geometries before and after the singularity (if a singular limit for the solutions of the wave equation with initial conditions specified away from the near-singular region is assumed to exist).

KEYWORDS: Penrose limit and pp-wave background, Spacetime Singularities.

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1. pp-waves

pp-wave geometries are an interesting family of essentially non-linear solutions of general relativity containing arbitrary functions of space-time coordinates and describing propagation of strong gravitational waves in space-time. Their importance extends beyond classical general relativity, since, in a large class of quantum gravitational theories (including perturbative string theories), they turn out to be unaffected by quantum corrections (see [1] and references therein). Furthermore, the presence of a light-like Killing vector in these backgrounds permits both a formulation of the matrix theory and an analytic solution of string theory sigma-models in the light-cone gauge.

It is interesting to consider the limit whereby the arbitrary functions contained in the pp-wave solutions develop isolated singularities. The corresponding light-like singularities in space-time geometry are reminiscent (even though different) from (space-like) cosmological singularities. Furthermore, the special standing of pp-waves in quantum gravitational theories makes these geometries a fruitful playground for studying quantum gravity in extreme high-curvature regime, and, in particular, exploring the possibilities to define dynamical transitions through space-time singularities.

In this note, we shall concentrate on the following class of simple pp-wave geometries:

$$ds^2 = -2dX^+ dX^- - \lambda F(X^+, \varepsilon) X^2 (dX^+)^2 + dX^2, \tag{1.1}$$

where $F(X^+, \varepsilon)$ is an arbitrary function developing a singularity at $X^+ = 0$ when ε is sent to 0 and λ is a number (the overall pp-wave profile normalization). We shall formally work in three space-time dimensions, but our derivations can be immediately extended to higher dimensions by replacing dX^2 with $dX^i dX^i$. We shall assume that F does not depend on

any dimensional parameters besides ε , in which case the $\varepsilon \rightarrow 0$ limit is scale-invariant, and on dimensional grounds, one can write F as

$$F(X^+, \varepsilon) = \frac{1}{\varepsilon^2} \Omega(X^+/\varepsilon). \tag{1.2}$$

Note that even though all our considerations will, strictly speaking, apply only to this “scale-invariant” case, one should expect that even if F depends on other dimensional parameters, at least in some cases, they should not affect the existence of the limit. Indeed, the singularity transition takes place in an ε -neighborhood of $X^+ = 0$ and it should not be particularly sensitive to dimensional parameters that stay finite as ε is taken to 0.

The case of $F(X^+, \varepsilon) \sim (1/X^+)^2$ has been previously studied¹ in [2]. However, our approach will be quite different, as we shall be considering the singular case as a limit of regularized geometries (rather than devising singularity transition recipes for the singular case itself). The motivation for this approach is that if the background (1.1) is used in the context of perturbative string theory (and related approaches to quantum gravity), it must satisfy Einstein’s equations (when those are exact, or else an appropriate generalization thereof), and this can only be ensured by working with a regularized space and taking an $\varepsilon \rightarrow 0$ limit in the end. Remarkably, it is precisely this limiting procedure that will be responsible for the emergence of discrete structures we are aiming to report.

We intend to study the wave equation for a free scalar field in the background (1.1). Whereas the wave equation itself does not describe the physical evolution on strongly curved spaces adequately, its solutions, the mode functions, are essential ingredients of any field-theoretical or string-theoretical set-up formulated in the background (1.1). Likewise, we shall not discuss how the scalar field should be coupled to the dilaton or p -form potentials necessary to make the metric (1.1) satisfy Einstein’s equations [2], and we shall assume that the scalar field is minimally coupled to the metric. One can hope that the robustness and genericity of the features we observe will make them survive at least for some modifications and extensions² of our present set-up. Our main observation will be that the $\varepsilon \rightarrow 0$ limit of the solutions to the wave equation in the background (1.1) with initial data specified away from the singularity typically exist only for discrete values of the parameter λ and only if the leading asymptotics of Ω in (1.2) are the same for X^+ going to $+\infty$ and $-\infty$. (The discreteness has been observed for a special class of pp-waves in [5], and our present objective is to show its generic nature.)

2. The wave equation

In the background (1.1), the Klein-Gordon wave equation takes the form

$$\partial_+ \partial_- \phi - \frac{1}{2} \partial_X^2 \phi - \frac{\lambda}{2} F(X^+, \varepsilon) X^2 \partial_-^2 \phi + \frac{m^2}{2} \phi = 0, \tag{2.1}$$

¹ Such scale-invariant $(1/X^+)^2$ dependencies of the pp-wave profile arise as Penrose limits of a broad class [3] of power-law space-time singularities (including the common singularities encountered in cosmology). This observation suggests (though not in a conclusive way) that it is natural to resolve such singularities in the scale-invariant fashion of (1.2).

²One natural problem to analyze is the evolution of a free quantum string in such regularized pp-waves and its singular limit [4] (see, e.g., [2] for related earlier work).

or, after Fourier-transforming,

$$\phi(X^+, X^-, X) = \frac{1}{\sqrt{2\pi}} \int dk_- \phi_{k_-}(X) \exp(ik_- X^-), \quad (2.2)$$

it can be re-written in the form

$$-i\dot{\phi} = -\frac{\partial_X^2 \phi}{2k_-} + \frac{\lambda k_-}{2} F(X^+, \varepsilon) X^2 \phi + \frac{m^2}{2k_-} \phi \quad (2.3)$$

(where the dot denotes the X^+ -derivative, and we have suppressed the k_- index on ϕ). The latter representation makes it manifest that the wave equation in pp-wave backgrounds takes the form of a Schrödinger equation (a well known fact, see for example [2]).

A general overview of singular limits in time-dependent Schrödinger equations, such as the $\varepsilon \rightarrow 0$ limit in (2.3), has been given in section 2 of [5]. It has been noted in particular that, if the Schrödinger equation possesses a finite-dimensional *dynamical group*, it reduces to a finite number of ordinary differential equations, which considerably simplifies the analysis of the singular limit. The equation (2.3) presents a particularly straightforward realization of this structure, since it is a Schrödinger equation for a linear quantum system, and as such, it can be reduced to ordinary differential equations (“classical equations of motion”) using the standard WKB techniques.

More specifically, one proceeds as follows. The formal “Hamiltonian” corresponding to (2.3) is

$$\mathcal{H} = \frac{P^2}{2k_-} + \frac{\lambda k_-}{2} F(t, \varepsilon) X^2 + \frac{m^2}{2k_-} \quad (2.4)$$

(where X^+ has been renamed to t in order to make the “quantum-mechanical” considerations look more familiar). Note that (2.4) is nothing but the Hamiltonian of a harmonic oscillator with a time-dependent frequency. The “Schrödinger” equation (2.3) is then solved by the ansatz

$$\phi(X_1, t_1 | X_2, t_2) = \mathcal{A}(t_1, t_2) \exp(-iS_{\text{cl}}[X_1, t_1 | X_2, t_2]), \quad (2.5)$$

where t_2 should be identified with X^+ of (2.3), if

$$S_{\text{cl}} = \int_{t_1}^{t_2} dt \left(P\dot{X} - \mathcal{H} \right) \Big|_{X=X_{\text{cl}}(X_1, t_1 | X_2, t_2)} \quad (2.6)$$

$$-2k_- \frac{\partial \mathcal{A}(t_1, t)}{\partial t} = \mathcal{A}(t_1, t) \frac{\partial^2 S_{\text{cl}}[X_1, t_1 | X, t]}{\partial X^2} \quad (2.7)$$

Here, S_{cl} is the “classical action” for the solution satisfying $X(t_1) = X_1$, $X(t_2) = X_2$. More general solutions to (2.3) are obtained by integrating (2.5) over X_1 , weighted by an arbitrary smooth wavepacket.

The classical equation of motion corresponding to (2.4) is

$$\ddot{X} + \lambda F(t, \varepsilon) X = 0 \quad (2.8)$$

Given two independent solutions to this equation, $f(t)$ and $h(t)$, one can straightforwardly construct the solution $X_{\text{cl}}(X_1, t_1|X_2, t_2)$ satisfying $X(t_1) = X_1$, $X(t_2) = X_2$, and subsequently evaluate the action (2.6):

$$S_{\text{cl}} = -\frac{k_-}{2} \frac{h_2 \dot{f}_1 - f_2 \dot{h}_1}{f_1 h_2 - h_1 f_2} X_1^2 + \frac{k_-}{2} \frac{f_1 \dot{h}_2 - h_1 \dot{f}_2}{f_1 h_2 - h_1 f_2} X_2^2 - k_- \frac{W[f, h]}{f_1 h_2 - h_1 f_2} X_1 X_2 - \frac{m^2}{2k_-} (t_2 - t_1), \quad (2.9)$$

where $W[f, h] = f\dot{h} - h\dot{f}$ is the Wronskian of $f(t)$ and $h(t)$ (independent of t), and we have introduced $f_1 = f(t_1), h_2 = h(t_2)$, etc.

With the above form of S_{cl} , (2.7) reduces to

$$\frac{\partial \mathcal{A}(t_1, t)}{\partial t} = -\frac{1}{2} \frac{f_1 \dot{h} - h_1 \dot{f}}{f_1 h - h_1 f} \mathcal{A}(t_1, t), \quad (2.10)$$

which can be solved as

$$\mathcal{A}(t_1, t_2) = \frac{\mathcal{N}}{\sqrt{f_1 h_2 - h_1 f_2}}, \quad (2.11)$$

with a normalization constant \mathcal{N} . Depending on how the solutions to the wave equation are to be used, different normalizations can be chosen. This ambiguity will not be relevant for our considerations, and it will be convenient to think of \mathcal{N} as being proportional to $\sqrt{W[f, h]}$, which makes independence on the normalization of f and h manifest.³ We thus arrive at the following complete basis of solutions (labelled by X_1) to the wave equation (2.3):

$$\phi \sim \frac{1}{\sqrt{\mathcal{C}(t_1, t_2)}} \exp\left(\frac{ik_- \partial_{t_1} \mathcal{C}}{2\mathcal{C}} X_1^2 - \frac{ik_- \partial_{t_2} \mathcal{C}}{2\mathcal{C}} X_2^2 + \frac{ik_-}{\mathcal{C}} X_1 X_2 + \frac{im^2}{2k_-} (t_2 - t_1)\right), \quad (2.12)$$

where we have introduced the ‘‘compression factor’’

$$\mathcal{C}(t_1, t_2) = \frac{f(t_1)h(t_2) - h(t_1)f(t_2)}{W[f, h]}. \quad (2.13)$$

Note that $\mathcal{C}(t_1, t_2)$ depends only on equation (2.8) and not on the choice of solutions f and h . Indeed, if one changes to a different solution basis

$$\begin{pmatrix} \tilde{f} \\ \tilde{h} \end{pmatrix} = A \begin{pmatrix} f \\ h \end{pmatrix}, \quad (2.14)$$

both numerator and denominator of (2.12) are multiplied by $\det A$. This property of $\mathcal{C}(t_1, t_2)$ makes it obvious that (2.12) is independent of the choice of f and h , as it should be. (Zeros of $\mathcal{C}(t_1, t_2)$ correspond to focal points of the equation (2.8), see appendix B of [5] for further details.) As a matter of fact, $\mathcal{C}(t_1, t_2)$ can be recognized as a solution (with respect to t_2) to (2.8) satisfying initial conditions

$$\mathcal{C}(t_1, t_2) \Big|_{t_2=t_1} = 0, \quad \partial_{t_2} \mathcal{C}(t_1, t_2) \Big|_{t_2=t_1} = 1. \quad (2.15)$$

³The choice of the branch structure of the square root in (2.11) is somewhat subtle but completely unambiguous and is given by the so called *Maslov phase* prescription. We shall refer the reader to appendix B of [5] for further details, which will not be relevant as far as our present goals are concerned.

3. The singular limit

The representation (2.12) for a complete basis of solutions to the wave equation (2.1) derived in the previous section is relevant for our present goals inasmuch as it relates our original problem to a simple ordinary differential equation (2.8). In particular, the existence of an $\varepsilon \rightarrow 0$ limit⁴ to (2.1) can be completely analyzed in terms of a single solution (2.15) to (2.8). (Note however, that if the $\varepsilon \rightarrow 0$ limit of (2.15) exists for all t_1 , the limit will exist for solutions satisfying arbitrary initial conditions. This follows from the fact that $\partial_{t_1} \mathcal{C}(t_1, t_2)$ satisfies initial conditions linearly independent of (2.15). The converse is also obviously true.) In this section, we shall analyze the $\varepsilon \rightarrow 0$ limit of $\mathcal{C}(t_1, t_2)$ for the special (“scale-invariant”) case of (1.2). We shall observe that the limit exists only for discrete values of the overall normalization of the pp-wave profile, and only if the leading asymptotics of Ω of (1.2) is the same for its argument going to plus and minus infinity.

3.1 Scaling

For the particular pp-wave given by (1.1)–(1.2), the “classical equation of motion” (2.8) takes the form

$$\ddot{X} + \frac{\lambda}{\varepsilon^2} \Omega(t/\varepsilon) X = 0, \tag{3.1}$$

and, as explained above, we are interested in the limiting behavior of the solution satisfying

$$X(t_1) = 0, \quad \dot{X}(t_1) = 1 \tag{3.2}$$

as ε is taken to 0.

The scaling properties of (1.2) permit rewriting this equation in a dimensionless form with $\eta = t/\varepsilon$, $Y(\eta) = X(\varepsilon\eta)$:

$$Y'' + \lambda\Omega(\eta)Y = 0, \quad Y|_{\eta=t_1/\varepsilon} = 0, \quad Y'|_{\eta=t_1/\varepsilon} = \varepsilon, \tag{3.3}$$

and, in this representation, one should be looking for an $\varepsilon \rightarrow 0$ limit of $Y(t/\varepsilon)$.

Note that the differential equation (3.3) is itself ε -independent, and the $\varepsilon \rightarrow 0$ limit has been translated into specifying initial conditions in the infinite past (we are assuming t_1 to be negative), while “observing” the results of the evolution at the point t/ε in the infinite future. In other words, the requirement that an $\varepsilon \rightarrow 0$ limit should exist has been translated into some constraints on the asymptotic behavior of solutions to an ε -independent differential equation. This latter formulation has a conspicuous flavor of a Sturm-Liouville problem, which makes the appearance of a discrete spectrum for λ hardly surprising. We shall see how this works out explicitly, after considering the asymptotic behavior of solutions to (3.3).

⁴We are speaking of an $\varepsilon \rightarrow 0$ limit corresponding to a meaningful dynamical evolution across the singularity. The typical situation is that $\mathcal{C}(t_1, t_2)$ blows up for $t_1 < 0$, $t_2 > 0$ in the $\varepsilon \rightarrow 0$ limit, i. e., the harmonic oscillator (2.8) is knocked out to infinity by the singular potential. In this case, the $\varepsilon \rightarrow 0$ limit of (2.12) exists in the mathematical sense, and is $\phi(t_2) = 0$ for $t_2 > 0$. This limit is, of course, completely meaningless as far as defining dynamical evolution across the singularity is concerned, and we refer to this situation as having no $\varepsilon \rightarrow 0$ limit.

3.2 Asymptotics

For the particular choices of $\Omega(\eta)$ we intend to consider, analyzing the asymptotic behavior of solutions to (3.3) is greatly simplified by the following lemma:

If the equation

$$Y'' + \Omega(\eta)Y = 0 \tag{3.4}$$

has solutions that behave for large η as⁵

$$Y_1(\eta) \sim \eta^a + o(\eta^a), \quad Y_2(\eta) \sim \eta^{1-a} + o(\eta^{1-a}) \tag{3.5}$$

with $a > 1/2$, then the equation

$$\tilde{Y}'' + \left(\Omega(\eta) + o(1/\eta^b)\right)\tilde{Y} = 0 \tag{3.6}$$

with $b > 2$ has solutions with the same asymptotic behavior.

(We shall be using the standard o - and O -symbols to describe the asymptotic behavior of our functions. $f(x)$ is said to be $o(g(x))$ at $x = \bar{x}$ if the limit of $f(x)/g(x)$ at $x = \bar{x}$ is 0, and $O(g(x))$ if the limit is finite.)

For the sake of brevity of notation, we shall phrase the proof for the asymptotics at $\eta = +\infty$. Take any solution to (3.6) and write it in the form

$$\tilde{Y}(\eta) = \xi(\eta)Y(\eta), \tag{3.7}$$

where $Y(\eta)$ is the solution to (3.4) satisfying

$$Y(\eta_0) = \tilde{Y}(\eta_0), \quad Y'(\eta_0) = \tilde{Y}'(\eta_0) \tag{3.8}$$

for some positive η_0 . Then $\xi(\eta)$ is a continuous function satisfying

$$\xi'' + 2\frac{Y'}{Y}\xi' + o(1/\eta^b)\xi = 0, \quad \xi(\eta_0) = 1, \quad \xi'(\eta_0) = 0, \tag{3.9}$$

which can be rewritten as

$$\eta^{-2a} (\eta^{2a}\xi')' = o(1/\eta)\xi' + o(1/\eta^b)\xi \tag{3.10}$$

(we have assumed that $Y(\eta)$ displays the dominant asymptotic t^a , rather than the subdominant asymptotic t^{1-a} , which should be generically possible to achieve for any $\tilde{Y}(\eta)$ by choosing η_0).

Let η_* be the first $\eta > \eta_0$ for which $|\xi'(\eta)| = A\eta^{-c}$, with $1 < c < \min(2a, b-1)$ and some positive constant A (if no such η_* exists, one can skip to (3.15) without any further considerations). Note that, by construction,

$$|\xi'(\eta)| < A\eta^{-c}, \quad |\xi(\eta)| < 1 + \frac{A}{(c-1)\eta_0^{c-1}}, \quad \text{for } \eta_0 < \eta < \eta_*. \tag{3.11}$$

⁵The relation between the powers of η in Y_1 and Y_2 is dictated by the conservation of the Wronskian. See also the discussion below (3.15).

One can integrate (3.10) from $\eta = \eta_0$ to $\eta = \eta_*$ to obtain

$$|\eta_*^{2a} \xi'(\eta_*)| \leq A \left| o(\eta^{2a-c}) \Big|_{\eta=\eta_0}^{\eta=\eta_*} \right| + \left(1 + \frac{A}{(c-1)\eta_0^{c-1}} \right) \left| o(\eta^{2a-b+1}) \Big|_{\eta=\eta_0}^{\eta=\eta_*} \right|, \quad (3.12)$$

which can be rewritten as (all the o -symbols are taken to be positive)

$$\eta_*^{2a-c} \leq (o(\eta_*^{2a-c}) + o(\eta_0^{2a-c})) + \left(\frac{1}{A} + \frac{1}{(c-1)\eta_0^{c-1}} \right) (o(\eta_*^{2a-b+1}) + o(\eta_0^{2a-b+1})) \quad (3.13)$$

or

$$1 \leq \left(\frac{o(\eta_*^{2a-c})}{\eta_*^{2a-c}} + \left(\frac{\eta_0}{\eta_*} \right)^{2a-c} \frac{o(\eta_0^{2a-c})}{\eta_0^{2a-c}} \right) + \left(\frac{1}{A} + \frac{1}{(c-1)\eta_0^{c-1}} \right) \left(\frac{o(\eta_*^{2a-b+1})}{\eta_*^{2a-c}} + \left(\frac{\eta_0}{\eta_*} \right)^{2a-c} \frac{o(\eta_0^{2a-b+1})}{\eta_0^{2a-c}} \right). \quad (3.14)$$

Since the right-hand side goes to 0 if η_* and η_0 go to infinity with $\eta_* > \eta_0$ and $1 < c < \min(2a, b-1)$ (as originally specified), it is possible to choose η_0 and A (independent of η_0) in such a way that, for all $\eta_* > \eta_0$, the inequality is *not* satisfied. Then no η_* (as defined above) exists, and instead of (3.11), one has

$$|\xi'(\eta)| < A\eta^{-c}, \quad \text{for } \eta > \eta_0. \quad (3.15)$$

Since $c > 1$, this implies that $\xi(\eta)$ goes to a (generically non-zero) constant at infinity,⁶ and the leading asymptotics of the solution $\tilde{Y}(\eta)$ is the same (up to a constant factor) as the leading asymptotics of $Y(\eta)$.

The existence and asymptotic behavior of the “subdominant” (η^{1-a}) solution can be inferred from the following consideration. The leading solution of (3.6) that we have just analyzed does not oscillate as η goes to $+\infty$. Hence, the equation (3.6) itself is non-oscillatory at $\eta = +\infty$ (see [6] for the relevant definitions and properties). Thus, it must have a solution (unique up to a constant) that grows slower than η^a at infinity (the “principal” solution), and by conservation of the Wronskian [6], this solution is

$$\tilde{Y}_2(\eta) \sim \tilde{Y}_1(\eta) \int_{\eta}^{+\infty} \frac{d\tilde{\eta}}{\tilde{Y}_1^2(\tilde{\eta})}, \quad (3.16)$$

with \tilde{Y}_1 being a dominant (“non-principal”) solution to (3.6). Since the asymptotic behavior of \tilde{Y}_1 has been established as

$$\tilde{Y}_1(\eta) \sim \eta^a + o(\eta^a), \quad (3.17)$$

together with (3.16), this gives

$$\tilde{Y}_2(\eta) \sim \eta^{1-a} + o(\eta^{1-a}). \quad (3.18)$$

This concludes the proof of the lemma.

⁶Since $|\xi(\eta) - 1| < A/(c-1)\eta_0^{c-1}$ with A independent of η_0 , one can always avoid the vanishing of $\xi(\eta)$ at infinity by choosing a sufficiently large value of η_0 .

3.3 Specific cases

Armed with the asymptotic behavior lemma, we can analyze the singular limit for specific pp-waves of interest. Namely, we shall assume that Ω of (1.2) behaves as

$$\Omega(\eta) = \frac{k_{\pm}}{\eta^2} + O(1/\eta^b) \tag{3.19}$$

for η going to $\pm\infty$ (respectively), for some constants k_{\pm} and $b > 2$.

We shall be interested in the case $0 \leq \lambda k_{\pm} < 1/4$. For these values, one can transform to the so-called Rosen co-ordinates, for which the $\varepsilon \rightarrow 0$ limit of the metric takes the form

$$ds^2 = -dX^+dX^- + (X^+)^{2(1-a_{\pm})} dX^2, \tag{3.20}$$

with a_{\pm} corresponding to $X^+ > 0$ and $X^+ < 0$ respectively,

$$a_{\pm} = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda k_{\pm}}. \tag{3.21}$$

The metric (3.20) is somewhat reminiscent of Friedman cosmologies (and arises from those in the Penrose limit [3]). The case $0 < \lambda k_{\pm} < 1/4$ has been termed “null-cosmology”. The case $k_{\pm} = 0$ refers to well-localized pp-waves, and, in the $\varepsilon \rightarrow 0$ limit, one is left with Minkowski space everywhere away from $X^+ = 0$, and a strong singularity at $X^+ = 0$. The case of the “light-like reflector plane” described in [5] falls precisely into this latter category.

By the asymptotic behavior lemma, the equation

$$Y'' + \lambda\Omega(\eta)Y = 0 \tag{3.22}$$

with Ω of the form (3.19) will have two solutions going as

$$Y_{1-}(\eta) = |\eta|^{a_-} + o(|\eta|^{a_-}), \quad Y_{2-}(\eta) = |\eta|^{1-a_-} + o(|\eta|^{1-a_-}) \tag{3.23}$$

for η approaching $-\infty$, and two solutions going as

$$Y_{1+}(\eta) = \eta^{a_+} + o(\eta^{a_+}), \quad Y_{2+}(\eta) = \eta^{1-a_+} + o(\eta^{1-a_+}) \tag{3.24}$$

for η approaching $+\infty$. The two sets of solutions are of course related:

$$\begin{pmatrix} Y_{1-} \\ Y_{2-} \end{pmatrix} = C(\lambda) \begin{pmatrix} Y_{1+} \\ Y_{2+} \end{pmatrix}, \tag{3.25}$$

where $C(\lambda)$ is a 2×2 -matrix.

Since we are interested in the evolution across the singular point $t = 0$, we choose an initial time $t_1 < 0$ and final time $t_2 > 0$. By (2.13),

$$\mathcal{C}(t_1, t_2) = \varepsilon \frac{Y_{1-}(t_1/\varepsilon)Y_{2-}(t_2/\varepsilon) - Y_{1-}(t_2/\varepsilon)Y_{2-}(t_1/\varepsilon)}{W_{\eta}[Y_{1-}, Y_{2-}]}, \tag{3.26}$$

where the Wronskian W_η is evaluated with respect to η . Using the asymptotic expansion of solutions in the past and future as given above together with (3.25) and $W_\eta[Y_{1-}, Y_{2-}] = 1 - 2a_-$, we obtain

$$\begin{aligned} \mathcal{C}(t_1, t_2) = & \frac{C_{11}(\lambda)}{2a_- - 1} |t_1|^{1-a_-} t_2^{a_+} \varepsilon^{a_- - a_+} + \frac{C_{12}(\lambda)}{2a_- - 1} |t_1|^{1-a_-} t_2^{1-a_+} \varepsilon^{a_- + a_+ - 1} \\ & - \frac{C_{21}(\lambda)}{2a_- - 1} |t_1|^{a_-} t_2^{a_+} \varepsilon^{1-a_- - a_+} - \frac{C_{22}(\lambda)}{2a_- - 1} |t_1|^{a_-} t_2^{1-a_+} \varepsilon^{a_+ - a_-}. \end{aligned} \tag{3.27}$$

We demand the $\varepsilon \rightarrow 0$ limit of (3.27) to exist for all fixed t_1 and t_2 . Simple inspection shows that the leading power of ε , i.e., the one that is the most negative, occurs in the term proportional to $C_{21}(\lambda)$, and that the associated power of ε is always negative. Thus, if the $\varepsilon \rightarrow 0$ limit exists, we must have

$$C_{21}(\lambda) = 0. \tag{3.28}$$

The power of ε in the term proportional to C_{12} is always positive, which makes it simply vanish in the limit. Furthermore, if $a_+ \neq a_-$ (i.e., $k_+ \neq k_-$), one must have either

$$C_{11}(\lambda) = 0 \tag{3.29}$$

or

$$C_{22}(\lambda) = 0 \tag{3.30}$$

(depending on the sign of $a_+ - a_-$).

Under generic conditions, (3.28) will leave only a discrete set of allowed values for λ . Indeed, (3.28) merely states that the solution to (3.22) subdominant at $\eta = -\infty$ does not receive any admixture of the dominant solution at $\eta = +\infty$, i.e. that the solution is subdominant at both $\pm\infty$. This is essentially a Sturm-Liouville problem, and the appearance of a discrete spectrum⁷ should be hardly surprising. A particular exactly solvable example for this discrete spectrum (there called “light-like reflector plane”) has been given in [5].

Imposing additionally (3.29) or (3.30) would make the determinant of $C(\lambda)$ vanish, in contradiction with the conservation of the Wronskian, and hence (3.22). We thus conclude that the $\varepsilon \rightarrow 0$ limit will exist for the solutions of the wave equation (in the class of pp-wave backgrounds we have been considering) if and only if λ belongs to a *discrete spectrum* of values, and the leading $(1/X^+)^2$ *asymptotics* of the pp-wave profile are *the same* on the both sides of the singularity.

4. Discreteness and matching conditions

In this note, we have considered the wave equation in families of pp-wave geometries developing strong scale-invariant singularities in certain limits. (The requirement of scale

⁷When $k_\pm = 0$, the subdominant solution to (3.22) is bounded at infinity. One can use the integral representation for such bounded solutions [7] to show that the two bounded solutions at $\pm\infty$ depend analytically on λ . The Wronskian of these two solutions (proportional to $C_{21}(\lambda)$) is then a non-identically-vanishing analytic function of λ . Hence its zeros, which are the solutions to (3.28), are discrete. For $k_\pm \neq 0$, some further analysis is needed to prove rigorously the discreteness of the spectrum for λ .

invariance is quite constraining, but one may expect that some of our results will extend to more general cases, since the presence of finite scale parameters is not likely to affect the dynamics in an infinitesimally small singular region.)

We have observed that the singular limit of the solutions to the wave equation with initial data specified away from the singularity exists only if:

- the absolute normalization of the pp-wave profile lies in a discrete spectrum (dependent on the specific way the singularity is resolved);
- the leading $(1/X^+)^2$ asymptotics of the pp-wave profile are the same before and after the singularity.

Even though these requirements are tremendously constraining, their predictive power is diminished in our present setting by the complete arbitrariness of the pp-wave profiles. One could speculate however, that if the time dependence of the pp-wave is governed by further specifications (as it might be, for example, if the pp-wave arises as a Penrose limit of some other geometry; see also footnote 1), the discrete features we observe will relate the structure of the singularity to some parameters of the space-time geometry away from the singular region.

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